Math 522 Exam 1 Solutions

1. Please calculate the number corresponding to the representation $(3,2,1)$, i.e. 321, in ten ways: in base 4,8,9,10,11,12,13,14,16,35.
   BONUS: repeat for the factoradic basis.

   $(321)_4 = 3\cdot 4^2 + 2\cdot 4 + 1 = (57)_{10}$ fifty-seven
   $(321)_8 = 3\cdot 8^2 + 2\cdot 8 + 1 = (209)_{10}$ two hundred nine
   $(321)_9 = 3\cdot 9^2 + 2\cdot 9 + 1 = (262)_{10}$ two hundred sixty-two
   $(321)_{10} = 3\cdot 10^2 + 2\cdot 10 + 1 = (321)_{10}$ three hundred twenty-one
   $(321)_{11} = 3\cdot 11^2 + 2\cdot 11 + 1 = (386)_{10}$ three hundred eighty-six
   $(321)_{12} = 3\cdot 12^2 + 2\cdot 12 + 1 = (457)_{10}$ four hundred fifty-seven
   $(321)_{13} = 3\cdot 13^2 + 2\cdot 13 + 1 = (534)_{10}$ five hundred thirty-four
   $(321)_{14} = 3\cdot 14^2 + 2\cdot 14 + 1 = (617)_{10}$ six hundred seventeen
   $(321)_{16} = 3\cdot 16^2 + 2\cdot 16 + 1 = (801)_{10}$ eight hundred one
   $(321)_{35} = 3\cdot 35^2 + 2\cdot 35 + 1 = (3746)_{10}$ thirty-seven hundred forty-six
   $(321)_F = 3\cdot 6 + 2\cdot 2 + 1 = (23)_{10}$ twenty-three

2. Let $(\ldots, b_3, b_2, b_1)$ be a basis. Let $(a_s, \ldots, a_1, a_0)$ and $(a'_s, \ldots, a'_1, a'_0)$ be two $(s+1)$-digit representations in that basis, corresponding to numbers A and A', respectively. Suppose that $a_i \geq a'_i$ for all $i \in [0, s]$. Prove that $A \geq A'$.

   METHOD 1: We prove this by induction on s. For $s = 0$, $A = a_0 \geq a'_0 = A'$, as desired. For $s > 0$, we split the numbers as follows: $A = C + D, A' = C' + D'$, where $C = a_s(b_s \cdots b_1), D = a_{s-1}(b_{s-1} \cdots b_1) + \cdots + a_1 b_1 + a_0, C' = a'_s(b_s \cdots b_1), D' = a'_{s-1}(b_{s-1} \cdots b_1) + \cdots a'_1 b_1 + a'_0$. Now, $C/C' = (a_s/a'_s) \geq 1$ [note $a'_s \neq 0$ because it's the leading digit], so $C \geq C'$. Now, D and D' have representations $(a_{s-1}, \ldots, a_0)$ and $(a'_{s-1}, \ldots, a'_0)$; hence, by the inductive hypothesis $D \geq D'$. Adding, we get $A = C + D \geq C' + D' = A'$, as desired.

   METHOD 2: We have $A - A' = (a_s(b_s \cdots b_1) + \cdots a_1 b_1 + a_0) - (a'_s(b_s \cdots b_1) + \cdots a'_1 b_1 + a'_0) = (a_s - a'_s)(b_s \cdots b_1) + \cdots (a_1 - a'_1)b_1 + (a_0 - a'_0) \geq 0$ since each summand is nonnegative, hence $A \geq A'$.

3. High score=103, Median score=75, Low (nonblank) score=55